JOURNAL OF APPROXIMATION THEORY 49, 289–299 (1987)

On the Rate of Convergence of Fourier Series for Functions of HBMV*

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Communicated by R. Bojanic

Received May 22, 1985

1. INTRODUCTION

F. John and L. Nirenberg [1] introduced the concept of BMO function in 1961 and D. Waterman [2] introduced the concept of ΛBV in 1972. Recently, X. L. Shi [3] introduced a new class, ΛBMV, between ΛBV and BMO, and applied it to the theory of Fourier series.

If $f \in L_{2\pi}$ and there exists a constant M so that for each $I = [a, b]$,

$$m_I(f) = |I|^{-1} \int_I |f(x) - f_I| \, dx < M, \tag{1.1}$$

where $|I| = b - a$, $f_I = |I|^{-1} \int_I f(x) \, dx$, then the function $f(x)$ is said to be a bounded mean oscillation function, written $f \in \text{BMO}$.

Let $\Lambda = \{\lambda_n\}$ be a nondecreasing sequence of positive numbers such that $\sum_{n=1}^{\infty} 1/\lambda_n = \infty$ and suppose $I_n = [a_n, b_n]$ is a sequence of nonoverlapping subintervals of $[0, 2\pi]$. If there exists an M so that, for each $I_n = [a_n, b_n]$,

$$\sum f(I_n)/\lambda_n < M,$$

where $f(I_n) = |f(b_n) - f(a_n)|$, then f is said to be a Λ-bounded variation function, denoted $f \in \Lambda\text{BV}$. In particular, if $\lambda_n = n$, we will let $\Lambda\text{BV} = \text{HBV}$ (harmonic bounded variation).

If for any real a and each sequence $I_n = [a_n, b_n]$ of nonoverlapping intervals on $[a, a + 2\pi]$,

$$M_\Lambda(f) := \sup_{\{I_n\}, a} \sum_{n=1}^{\infty} m_{I_n}(f)/\lambda_n < +\infty,$$

* Project supported by the Science Fund of the Chinese Academy of Sciences.

0021-9045/87 $3.00

holds, then f is said to be Λ-bounded mean variation function, written $f \in \Lambda\mathrm{BMV}$.

X. L. Shi proved

(i) $\Lambda\mathrm{BV} \subsetneqq \Lambda\mathrm{BMV} \subsetneqq \mathrm{BMO}$.

(ii) If $f \in \mathrm{HBMV}$, then the partial sum of Fourier series of f $S_n(x)$ converges to $S(x) = \frac{1}{2}(f(x+0) + f(x-0))$ at the points where $S(x)$ is well defined and converges uniformly on closed intervals of continuity points of f.

(iii) Let $f \in \Lambda\mathrm{BMV}$ and for $I = [a, b]$ denote

$$M_\Lambda(f, I) = \sup_{I_n \subset I} \sum_{n=1}^{\infty} m_{I_n}(f)/\lambda_n.$$

If $x_0 \in (a, b)$ is a continuity point of f, then

$$\lim_{y \to +0} M_\Lambda(f, [x_0 - y, x_0 + y]) = 0.$$

On the other hand, an estimate on the rate of convergence of Fourier series for functions of bounded variation was first established by R. Bojanic [4]. Later R. Bojanic and D. Waterman (see [5]) considered the class between two extremes, BV and HBV, by setting $\Lambda = \{n^s\}$ $(0 < s < 1)$. Recently, D. Waterman [6] proved that if $\{\lambda_k/k\}$ is nonincreasing and $f \in \Lambda\mathrm{BV}$ and $\pi/(n+1) < a_n < a_{n+1} < \cdots < a_0 = \pi$, then

$$\left| S_n(x) - \frac{1}{2}[f(x+0) + f(x-0)] \right|$$
$$\leqslant (1 + 1/\pi)\frac{\lambda_{n+1}}{n+1} V(\pi) + \frac{\pi}{n+1} \sum_{i=0}^{n-1} V(a_i)[H(a_{i+1}) - H(a_i)] \qquad (1.2)$$

where $V(t) = V_\Lambda(g_x, [0, t])$, $g_x(t) = f(x+t) + f(x-t) - f(x+0) - f(x-0)$ and $H(t)$ is a continuous nonincreasing function on $(0, \pi]$ such that

$$H(t) = \lambda_k/t, \qquad \text{for} \quad t = k\pi/(n+1),\ k = 1, ..., n+1.$$

The estimate (1.2) is not adequate for $f \in \mathrm{HBV}$. Hence, for the extreme HBV, the question of estimate of the rate of convergence remains open to this day. In this paper we will solve the question for larger class HBMV. As a special case, we obtain the rate for $f \in \mathrm{HBV}$. We define so-called pointwise modulus of continuity at x as follows:

$$w(g; x, t) := \sup_{|y-x| \leqslant t,\, y} |g(y) - g(x)|.$$

It is clear that if $f \in \Lambda\mathrm{BV}$, then

$$w(g_x; 0, t) \leqslant V_\Lambda(g_x, [0, t]) = V(t). \tag{1.3}$$

2. Results

Now we state our results.

THEOREM 2.1. *Let* $f \in \Lambda\mathrm{BMV}$; *then*

$$\left| S_n(x) - \frac{1}{2}(f(x+0) + f(x-0)) \right|$$
$$\leqslant 3w(7\pi/(2n+1)) + 2\pi M_\Lambda(\pi) \Big/ \sum_{k=1}^{[n/2]} 1/\lambda_k$$
$$+ 2\pi \sum_{k=1}^{[n/2]} M_\Lambda((4k+3)\pi/(2n+1)) \Big/ \left(k \sum_{i=1}^{k} 1/\lambda_i\right), \tag{2.1}$$

where $w(t) = w(g_x, 0, t)$ *and* $M_\Lambda(t) = M_\Lambda(g_x, [0, t])$ *and* $g_x(t) = f(x+t) + f(x-t) - f(x+0) - f(x-0)$.

THEOREM 2.2. *Let* $f \in \mathrm{HBMV}$; *then we have*

$$\left| S_n(x) - \frac{1}{2}(f(x+0) + f(x-0)) \right|$$
$$\leqslant 3w(7\pi/(2n+1)) + 2\pi M_H((4n\mathscr{E}_n + 3)\pi/(2n+1))$$
$$+ 2\pi M_H(\pi) \Big/ \sum_{i=1}^{[n/2]} 1/\lambda_i$$
$$+ 2\pi \sum_{k=[n\mathscr{E}_n]+1}^{[n/2]-1} M_H((4k+3)\pi/(2n+1)) \Big/ \left(k \sum_{i=1}^{k} 1/\lambda_i\right), \tag{2.2}$$

where $\mathscr{E}_n \downarrow 0$ *and* $\log \mathscr{E}_n = o(\log n)$.
Set $\mathscr{E}_n = (\log n)^{-1}$; *we have*

COROLLARY 2.3. *For* $f \in \mathrm{HBMV}$,

$$S_n(x) - \tfrac{1}{2}(f(x+0) + f(x-0))$$
$$= 0(w(7\pi/(2n+1) + M_H(4\pi/\log n) + (\log\log n/\log n)\, M_H(\pi)).$$

Noting (1.3), we obtain

COROLLARY 2.4. *For* $f \in \mathrm{HBV}$,

$$S_n(x) - \tfrac{1}{2}(f(x+0) + f(x-0)) = 0(V_H(4\pi/\log n) + (\log\log n/\log n)\, V_H(\pi)).$$

3. PROOFS

Proof of Theorem 2.1. We have

$$\begin{aligned}
\Delta_n(x) &= S_n(x) - \frac{1}{2}(f(x+0) + f(x-0)) \\
&= \frac{1}{2\pi}\int_0^{\pi} g_x(t)\,\frac{\sin(n+\frac{1}{2})\,t}{\sin\frac{1}{2}t}\,dt \\
&= \frac{1}{2\pi}\int_0^{\pi/(2n+1)} g_x(t)\,\frac{\sin(n+\frac{1}{2})\,t}{\sin\frac{1}{2}t}\,dt \\
&\quad + \frac{1}{2\pi}\sum_{k=1}^{2n}\int_0^{\pi/(2n+1)} g_x\left(t+\frac{k\pi}{2n+1}\right)\frac{\sin((n+\frac{1}{2})\,t+\frac{1}{2}k\pi)}{\sin(\frac{1}{2}t+k\pi/(4n+2))}\,dt \\
&= \frac{1}{2\pi}\int_0^{\pi/(2n+1)} g_x(t)\,\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}\,dt \\
&\quad + \frac{1}{2\pi}\int_0^{\pi/(2n+1)}\left\{\sum_{k=1}^{n} g_x\left(t+\frac{(2k-1)\,\pi}{2n+1}\right)\frac{(-1)^{k-1}\cos(n+\frac{1}{2})\,t}{\sin(\frac{1}{2}t+(2k-1)\pi/(4n+2))}\right. \\
&\quad \left.+ \sum_{k=1}^{n} g_x\left(t+\frac{2k\pi}{2n+1}\right)\frac{(-1)^k\sin(n+\frac{1}{2})\,t}{\sin(\frac{1}{2}t+2k\pi/(4n+2))}\,dt\right\} \\
&:= R_1 + R_2.
\end{aligned} \tag{3.1}$$

It is obvious that

$$R_1 \leqslant \tfrac{1}{2}w(\pi/(2n+1)). \tag{3.2}$$

If n is odd, set $n = 2m+1$, then

$$\begin{aligned}
2\pi R_2 = \int_0^{\pi/(2n+1)} &\left[g_x\left(t+\frac{\pi}{2n+1}\right)\frac{\cos(n+\frac{1}{2})\,t}{\sin(\frac{1}{2}t+\pi/(4n+2))}\right. \\
&\left. - g_x\left(t+\frac{2\pi}{2n+1}\right)\frac{\sin(n+\frac{1}{2})\,t}{\sin(\frac{1}{2}t+2\pi/(4n+2))}\right] dt
\end{aligned}$$

$$+\int_0^{\pi/(2n+1)}\left\{\cos\left(n+\frac{1}{2}\right)t\right.$$

$$\times\sum_{k=1}^{m}\left[g_x\left(t+\frac{4k+1}{2n+1}\pi\right)\sin^{-1}\frac{1}{2}\left(t+\frac{4k+1}{2n+1}\pi\right)\right.$$

$$\left.-g_x\left(t+\frac{4k-1}{2n+1}\pi\right)\sin^{-1}\frac{1}{2}\left(t+\frac{4k-1}{2n+1}\pi\right)\right]$$

$$+\sin\left(n+\frac{1}{2}\right)t\cdot\sum_{k=1}^{m}\left[g_x\left(t+\frac{4k}{2n+1}\pi\right)\sin^{-1}\frac{1}{2}\left(t+\frac{4k}{2n+1}\pi\right)\right.$$

$$\left.\left.-g_x\left(t+\frac{4k+2}{2n+1}\pi\right)\sin^{-1}\frac{1}{2}\left(t+\frac{4k+2}{2n+1}\pi\right)\right]\right\}dx$$

$$:=2\pi(R_{21}+R_{22}). \tag{3.3}$$

It is clear that

$$|R_{21}|\leqslant\tfrac{3}{4}w(3\pi/(2n+1)). \tag{3.4}$$

Rewrite that

$$2\pi R_{22}=\int_0^{\pi/(2n+1)}\left\{\cos\left(n+\frac{1}{2}\right)t\sum_{k=1}^{m}\left[g_x\left(t+\frac{4k+1}{2n+1}\pi\right)\right.\right.$$

$$\left.-g_x\left(t+\frac{4k-1}{2n+1}\pi\right)\right]\sin^{-1}\frac{1}{2}\left(t+\frac{4k-1}{2n+1}\pi\right)$$

$$+\sin\left(n+\frac{1}{2}\right)t\sum_{k=1}^{m}\left[g_x\left(t+\frac{4k}{2n+1}\pi\right)-g_x\left(t+\frac{4k+2}{2n+1}\pi\right)\right]$$

$$\left.\times\sin^{-1}\frac{1}{2}\left(t+\frac{4k}{2n+1}\pi\right)\right\}dt$$

$$+\int_0^{\pi/(2n+1)}\left\{\cos\left(n+\frac{1}{2}\right)t\sum_{k=1}^{m}g_x\left(t+\frac{4k+1}{2n+1}\pi\right)\right.$$

$$\times\left[\sin^{-1}\frac{1}{2}\left(t+\frac{4k+1}{2n+1}\right)-\sin^{-1}\frac{1}{2}\left(t+\frac{4k-1}{2n+1}\right)\right]$$

$$+\sin\left(n+\frac{1}{2}\right)t\sum_{k=1}^{m}g_x\left(t+\frac{4k+2}{2n+1}\pi\right)$$

$$\left.\times\left[\sin^{-1}\frac{1}{2}\left(t+\frac{4k}{2n+1}\pi\right)-\sin^{-1}\frac{1}{2}\left(t+\frac{4k+2}{2n+1}\pi\right)\right]\right\}dt$$

$$:=2\pi(R'_{22}+R''_{22}). \tag{3.5}$$

It is obvious that

$$|R'_{22}| \leqslant \frac{2n+1}{6\pi} \int_0^{\pi/(2n+1)} \sum_{k=1}^{m} \left(\left| g_x\left(t+\frac{4k-1}{2n+1}\pi\right) - g_x\left(t+\frac{4k+1}{2n+1}\pi\right) \right| \Big/ k + \left| g_x\left(t+\frac{4k}{2n+1}\pi\right) - g_x\left(t+\frac{4k+2}{2n+1}\pi\right) \right| \Big/ k \right) dx. \tag{3.6}$$

Applying the Abel transformation to R''_{22}, we obtain

$$\begin{aligned}
|R''_{22}| = \Bigg| &\frac{1}{\pi} \sin\frac{\pi}{4n+2} \int_0^{\pi/(2n+1)} \Bigg[-\cos\left(n+\frac{1}{2}\right) t g_x\left(t+\frac{5\pi}{2n+1}\right) \\
&\times \sum_{k=1}^{m} \frac{\cos\frac{1}{2}(t+4k\pi/(2n+1))}{\sin\frac{1}{2}(t+(4k-1)\pi/(2n+1)) \sin\frac{1}{2}(t+(4k+1)\pi/(2n+1))} \\
&+ \sin\left(n+\frac{1}{2}\right) t g_x\left(t+\frac{6\pi}{2n+1}\right) \\
&\times \sum_{k=1}^{m} \frac{\cos\frac{1}{2}(t+(4k+1)\pi/(2n+1))}{\sin\frac{1}{2}(t+4k\pi/(2n+1)) \sin\frac{1}{2}(t+(4k+2)\pi/(2n+1))} \Bigg] dt \\
&+ \frac{1}{\pi} \sin\frac{\pi}{4n+2} \int_0^{\pi/(2n+1)} \Bigg\{ -\cos\left(n+\frac{1}{2}\right) t \\
&\times \sum_{k=2}^{m} \left[g_x\left(t+\frac{4k+1}{2n+1}\pi\right) - g_x\left(t+\frac{4k-3}{2n+1}\pi\right) \right] \\
&\times \sum_{i=k}^{m} \frac{\cos\frac{1}{2}(t+(4i\pi/(2n+1)))}{\sin\frac{1}{2}(t+(4i-1)\pi/(2n+1)) \sin\frac{1}{2}(t+(4i+1)\pi/(2n+1))} \\
&+ \sin\left(n+\frac{1}{2}\right) t \sum_{k=2}^{m} \left[g_x\left(t+\frac{4k+2}{2n+1}\pi\right) - g_x\left(t+\frac{4k-2}{2n+1}\pi\right) \right] \\
&\times \sum_{i=k}^{m} \frac{\cos\frac{1}{2}(t+(4i+1)\pi/(2n+1))}{\sin\frac{1}{2}(t+4i\pi/(2n+1)) \sin\frac{1}{2}(t+(4i+2)\pi/(2n+1))} \Bigg\} dt \Bigg| \\
\leqslant\, & w(7\pi/(2n+1)) + \int_0^{\pi/(2n+1)} \sum_{k=2}^{m} \left| g_x\left(t+\frac{4k+1}{2n+1}\pi\right) \right. \\
&\left. - g_x\left(t+\frac{4k-3}{2n+1}\pi\right) \right| \sum_{i=k}^{m} \frac{2n+1}{(4i-1)(4i+1)} dt \\
&+ \int_0^{\pi/(2n+1)} \sum_{k=2}^{m} \left| g_x\left(t+\frac{4k+2}{2n+1}\pi\right) - g_x\left(t+\frac{4k-2}{2n+1}\pi\right) \right| \\
&\times \sum_{i=k}^{m} \frac{2n+1}{(4i)(4i+2)} dt
\end{aligned}$$

$$\leqslant w(7\pi/(2n+1))+\frac{2n+1}{6}$$

$$\times\int_0^{\pi/(2n+1)}\left\{\sum_{k=2}^{m}k^{-1}\left|g_x\left(t+\frac{4k+1}{2n+1}\pi\right)-g_x\left(t+\frac{4k-3}{2n+1}\pi\right)\right|\right.$$

$$\left.+\sum_{k=2}^{m}k^{-1}\left|g_x\left(t+\frac{4k+2}{2n+1}\pi\right)-g_x\left(t+\frac{4k-2}{2n+1}\pi\right)\right|\right\}dt$$

$$\leqslant w(7\pi/(2n+1))+\frac{2n+1}{6}$$

$$\times\int_0^{\pi/(2n+1)}\left\{\sum_{k=2}^{m}k^{-1}\left|g_x\left(t+\frac{4k+1}{2n+1}\pi\right)-g_x\left(t+\frac{4k-1}{2n+1}\pi\right)\right|\right.$$

$$\left.+\sum_{k=2}^{m}k^{-1}\left|g_x\left(t+\frac{4k+2}{2n+1}\pi\right)-g_x\left(t+\frac{4k}{2n+1}\pi\right)\right|\right\}dt$$

$$+\frac{2n+1}{6}\int_0^{\pi/(2n+1)}\left\{\sum_{k=2}^{m}k^{-1}\left|g_x\left(t+\frac{4k-1}{2n+1}\pi\right)-g_x\left(t+\frac{4k-3}{2n+1}\pi\right)\right|\right.$$

$$\left.+\sum_{k=2}^{m}k^{-1}\left|g_x\left(t+\frac{4k}{2n+1}\pi\right)-g_x\left(t+\frac{4k-2}{2n+1}\pi\right)\right|\right\}dt. \tag{3.7}$$

Then from (3.5)–(3.7),

$$|R_{22}|\leqslant w(7\pi/(2n+1))+\frac{2}{9}(2n+1)$$

$$\times\int_0^{\pi/(2n+1)}\sum_{k=1}^{m}k^{-1}\left\{\left|g_x\left(t+\frac{4k-1}{2n+1}\pi\right)-g_x\left(t+\frac{4k+1}{2n+1}\pi\right)\right|\right.$$

$$\left.+\left|g_x\left(t+\frac{4k\pi}{2n+1}\right)-g_x\left(t+\frac{4k+2}{2n+1}\pi\right)\right|\right\}dt+\frac{2n+1}{6}$$

$$\times\int_0^{\pi/(2n+1)}\sum_{k=2}^{m}k^{-1}\left\{\left|g_x\left(t+\frac{4k-1}{2n+1}\pi\right)-g_x\left(t+\frac{4k-3}{2n+1}\pi\right)\right|\right.$$

$$\left.+\left|g_x\left(t+\frac{4k\pi}{2n+1}\right)-g_x\left(t+\frac{4k-2}{2n+1}\pi\right)\right|\right\}dt \tag{3.8}$$

$$=w(7\pi/(2n+1))+\frac{2(2n+1)}{9m}$$

$$\times\int_0^{\pi/(2n+1)}\left\{\sum_{k=1}^{m}\left|g_x\left(t+\frac{4k-1}{2n+1}\pi\right)-g_x\left(t+\frac{4k+1}{2n+1}\pi\right)\right|\right.$$

$$\left.+\left|g_x\left(t+\frac{4k}{2n+1}\pi\right)-g_x\left(t+\frac{4k+2}{2n+1}\pi\right)\right|\right\}dt+\frac{2n+1}{6m}$$

$$\times \int_0^{\pi/(2n+1)} \sum_{k=2}^{m} \left\{ \left| g_x\left(t+\frac{4k-1}{2n+1}\pi\right) - g_x\left(t+\frac{4k-3}{2n+1}\pi\right) \right| \right.$$

$$\left. + \left| g_x\left(t+\frac{4k}{2n+1}\pi\right) - g_x\left(t+\frac{4k-2}{2n+1}\pi\right) \right| \right\} dt$$

$$+ \frac{2(2n+1)}{9} \int_0^{\pi/(2n+1)} \sum_{k=1}^{m-1} k^{-2} \sum_{i=1}^{k} \left\{ \left| g_x\left(t+\frac{4i-1}{2n+1}\pi\right) \right. \right.$$

$$\left. - g_x\left(t+\frac{4i+1}{2n+1}\pi\right) \right|$$

$$\left. + g_x\left(t+\frac{4i\pi}{2n+1}\right) - g_x\left(t+\frac{4i+2}{2n+1}\pi\right) \right| \right\} dt + \frac{2n+1}{6}$$

$$\times \int_0^{\pi/(2n+1)} \sum_{k=2}^{m-1} k^{-2} \sum_{i=2}^{k} \left\{ \left| g_x\left(t+\frac{4i-1}{2n+1}\pi\right) - g_x\left(t+\frac{4i-3}{2n+1}\right) \right| \right.$$

$$\left. + \left| g_x\left(t+\frac{4i\pi}{2n+1}\right) - g_x\left(t+\frac{4i-2}{2n+1}\pi\right) \right| \right\} dt.$$

Denote

$$I_k = \left[\frac{4k-1}{2n+1}\pi, \frac{4k+3}{2n+1}\pi\right], \qquad k = 1, 2, \ldots, m$$

$$I'_k = \left[\frac{4k-3}{2n+1}\pi, \frac{4k+1}{2n+1}\pi\right], \qquad k = 2, \ldots, m.$$

We have

$$|R_{22}| \leqslant w(7\pi/(2n+1)) + \frac{2(2n+1)}{9m} \sum_{k=1}^{m} \int_{((4k-1)/(2n+1))\pi}^{((4k+3)/(2n+1))\pi} |g_x(u) - g_{I_k}|\, du$$

$$+ \frac{2n+1}{6m} \sum_{k=2}^{m} \int_{(4k-3)/(2n+1))\pi}^{((4k+1)/(2n+1))\pi} |g_x(u) - g_{I'_k}|\, du$$

$$+ \frac{2(2n+1)}{9} \sum_{k=1}^{m-1} k^{-2} \sum_{i=1}^{k} \int_{((4i-1)/(2n+1))\pi}^{((4i+3)/(2n+1))\pi} |g_x(u) - g_{I_k}|\, du$$

$$+ \frac{2n+1}{6} \sum_{k=2}^{m-1} k^{-2} \sum_{i=2}^{k} \int_{((4i-3)/(2n+1))\pi}^{((4i+1)/(2n+1))\pi} |g_x(u) - g_{I'_k}|\, du$$

$$\leqslant w(7\pi/(2n+1)) + \frac{\pi}{m} \sum_{k=1}^{m} m_{I_k}(g_x) + \frac{2\pi}{3m} \sum_{k=2}^{m} m_{I'_k}(g_x)$$

$$+ \pi \sum_{k=1}^{m-1} k^{-2} \sum_{i=1}^{k} m_{I_i}(g_x) + \frac{2}{3}\pi \sum_{k=2}^{m-1} k^{-2} \sum_{i=2}^{k} m_{I'_i}(g_x).$$

Arranging $\{m_{I_k}(g_x)\}$, $\{m_{I'_k}(g_x)\}$, $\{m_{I_i}(g_x)\}$, $\{m_{I'_i}(g_x)\}$ in nonincreasing order and applying Chebyshev inequality [7], we have

$$\begin{aligned}
|R_{22}| &\leqslant w(7\pi/(2n+1)) + \pi M_A(\pi) \Big/ \left(\sum_{i=1}^{m} 1/\lambda_i \right) \\
&\quad + \frac{2}{3}\pi M_A(\pi) \Big/ \left(\sum_{i=1}^{m} 1/\lambda_i) \right) \\
&\quad + \pi \sum_{k=1}^{m-1} M_A((4k+3)\,\pi/(2n+1)) \Big/ \left(k \sum_{i=1}^{k} 1/\lambda_i \right) \\
&\quad + \frac{2}{3}\pi \sum_{k=2}^{m-1} M_A((4k+1)\,\pi/(2n+1)) \Big/ \left(k \sum_{i=1}^{k} 1/\lambda_i \right) \\
&\leqslant w(7\pi/(2n+1)) + 2\pi M_A(\pi) \Big/ \left(\sum_{i=1}^{m} 1/\lambda_i \right) \\
&\quad + 2\pi \sum_{k=1}^{m-1} M_A((4k+3)\,\pi/(2n+1)) \Big/ \left(k \sum_{i=1}^{k} 1/\lambda_i \right) \qquad (3.9)
\end{aligned}$$

Finally, combining (3.1)–(3.4) and (3.9), (2.1) follows for odd n. For even n (2.1) can be similarly proved. The proof is completed.

Proof of Theorem 2.2. Let n be odd. From (3.8) we have

$$\begin{aligned}
|R_{22}| &\leqslant w(7\pi/(2n+1)) \\
&\quad + \left\{ \frac{2}{9}(2n+1) \int_0^{\pi/(2n+1)} \sum_{k=1}^{[n\delta_n]} k^{-1} \left[\left| g_x\left(t + \frac{4k-1}{2n+1}\pi\right) \right.\right.\right. \\
&\quad \left. - g_x\left(t + \frac{4k+1}{2n+1}\pi\right) \right| + \left| g_x\left(t + \frac{4k}{2n+1}\pi\right) - g_x\left(t + \frac{4k+2}{2n+1}\pi\right) \right| \Bigg] dt \\
&\quad + \frac{2n+1}{6} \int_0^{\pi/(2n+1)} \sum_{k=2}^{[n\delta_n]} k^{-1} \left[\left| g_x\left(t + \frac{4k-1}{2n+1}\pi\right) - g_x\left(t + \frac{4k-3}{2n+1}\pi\right) \right| \right. \\
&\quad \left.\left. + g_x\left(t + \frac{4k}{2n+1}\pi\right) - g_x\left(t + \frac{4k-2}{2n+1}\pi\right) \right| \right] dt \Bigg\} \\
&\quad + \left\{ \frac{2}{9}(2n+1) \int_0^{\pi/(2n+1)} \sum_{k=[n\delta_n]+1}^{m} k^{-1} \left[\left| g_x\left(t + \frac{4k-1}{2n+1}\pi\right) \right.\right.\right. \\
&\quad \left. - g_x\left(t + \frac{4k+1}{2n+1}\pi\right) \right| + \left| g_x\left(t + \frac{4k}{2n+1}\pi\right) - g_x\left(t + \frac{4k+2}{2n+1}\pi\right) \right| \Bigg] dt
\end{aligned}$$

$$+\frac{2n+1}{6}\int_0^{\pi/(2n+1)} \sum_{k=[n\mathscr{E}_n]+1}^{m} k^{-1}\left[\left|g_x\left(t+\frac{4k-1}{2n+1}\pi\right)\right.\right.$$

$$\left.-g_x\left(t+\frac{4k-3}{2n+1}\pi\right)\right|+\left|g_x\left(t+\frac{4k}{2n+1}\pi\right)-g_x\left(t+\frac{4k-2}{2n+1}\pi\right)\right|\Bigg]\,dt\Bigg\}$$

$$:= w(7\pi/(2n+1)) + r'_{22} + r''_{22}. \tag{3.10}$$

For r'_{22}, we have

$$r'_{22} \leqslant \pi \sum_{k=1}^{[n\mathscr{E}_n]} m_{I_k}(g_x)/k + \frac{2}{3}\pi \sum_{k=1}^{[n\mathscr{E}_n]} m_{I'_k}(g_x)/k$$

$$\leqslant \frac{5\pi}{3} M_H((4[n\mathscr{E}_n]+3)\,\pi/(2n+1)). \tag{3.11}$$

For r''_{22} applying the Abel transformation and the method similar to that in proof of Theorem 2.1, we obtain

$$r''_{22} \leqslant \frac{5}{3}\pi M_H(\pi)\Bigg/\left(\sum_{i=1}^{[n/2]} 1/\lambda_i\right)$$

$$+\frac{14}{9}\pi \sum_{k=[n\mathscr{E}_n]+1}^{[n/2]-1} M_H((4k+3)\,\pi/(2n+1))\Bigg/\left(k\sum_{i=1}^{k} 1/\lambda_i\right). \tag{3.12}$$

From (3.10)–(3.12) it follows that

$$|R_{22}| \leqslant w(7\pi/(2n+1)) + \frac{5}{3}\pi M_H((4[n\mathscr{E}_n]+3)\,\pi/(2n+1))$$

$$+\frac{5}{3}\pi M_H(\pi)\Bigg/\left(\sum_{i=1}^{[n/2]} 1/\lambda_i\right)$$

$$+\frac{14}{9}\pi \sum_{k=[n\mathscr{E}_n]+1}^{[n/2]-1} M_H((4k+3)\,\pi/(2n+1))\Bigg/\left(k\sum_{i=1}^{k} 1/\lambda_i\right). \tag{3.13}$$

Combining (3.1)–(3.4) and (3.13), we obtain (2.2). For even n (2.2) holds still. The proof of Theorem 2.2 is completed.

References

1. F. John and L. Nirenberg, On functions of bounded mean oscillation, *Comm. Pure Appl. Math.* **14** (1961), 415–426.
2. D. Waterman, On convergence of Fourier series of functons of generalized bounded variation, *Studia Math.* **44** (1972), 107–117.

3. X. L. SHI, On ΛBMV functions with some applications to theory of Fourier series, *Sci. Sinica, Ser. A* **28** (1985), 147–158.
4. R. BOJANIC, An estimate of the rate of convergence for Fourier series of bounded variation, *Publ. Inst. Math. (Beograd)* **26**, No. 40 (1979), 57–60.
5. R. BOJANIC AND D. WATERMAN, On the rate of convergence of Fourier series of functions of generalized bounded variation, *Akad. Nauka Umjet. Bosne Hercegov. Rad.* **74** (1983), 5–11.
6. D. WATERMAN, Estimating functions by partial suns of their Fourier series, *J. Math. Anal. Appl.* **87** (1982), 51–57.
7. D. S. MITRINOVIC, "Analytic Inequalities," Springer–Verlag, Berlin, 1970.